Automatic Theorem Proving

A Very Brief Introduction

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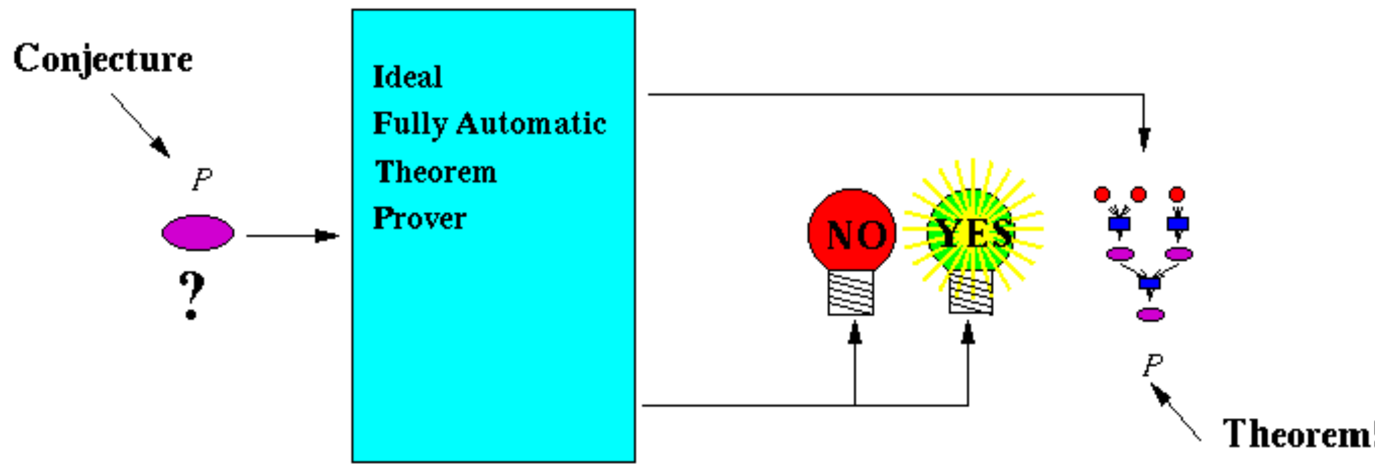
Definition

- Automated theorem proving (also known as ATP or automated deduction) is a subfield of automated reasoning and mathematical logic dealing with proving mathematical theorems by computer programs.

Source: Wikipedia

The Dream

Zzzz



source: http://www.cs.utexas.edu/users/moore/

Good News

- First-order logic together with set theory is expressive enough to serve as a foundation for mathematics
  - Frege, Whitehead, Russel
  - First-order logic consists of predicates, quantifiers, variables, and logical connectives, e.g.

$$\forall X, Y[\text{mother}(X,Y) \text{ if } \text{parent}(X,Y) \wedge \text{female}(X)]$$

More Good News

- First-order logic is sound and complete – Goedel
  - For any finite first-order theory *T* and any sentence *s* in the language of the theory, there is a formal proof of *s* in *T* if and only if *s* is satisfied by every model of *T*,

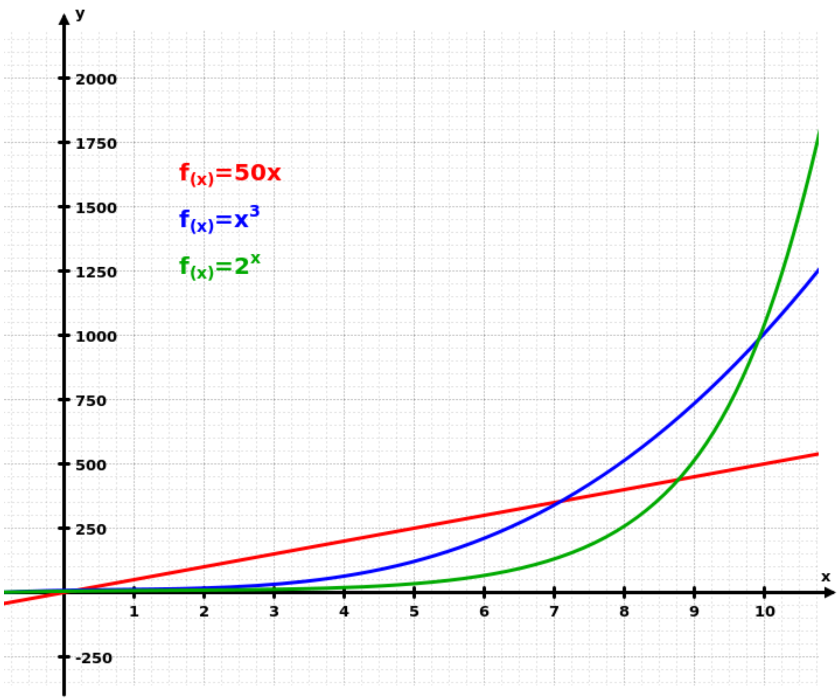
$$\text{For all models } M \text{ of some theory } T,\ T \vdash s \text{ iff } M \models s$$

Some Bad News

- First-order logic is *semi-decidable* – Church/ Turing
  - Given some decision procedure P:
    - P will accept and return a proof for some sentence s if s is valid.
    - P can reject or *loop forever* if s is *not* valid.

➔The first blow to our dream!

More Bad News



- Any decision procedure P given some valid sentence s runs at best in *NP* time.
  - That is, the time it takes to run P grows exponentially with the complexity of the sentence s.

➔ The second blow to our dream!

Problem

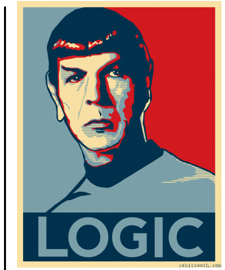
- If an ATP runs a long time you don’t know if the cause of this is the undecidability problem or the NP problem.

Goedel's Incompleteness Theorem

- Even though first-order logic is sound and complete there are some domains that are not finitely axiomatizable – that is there are no finite theories that describe this domain,
  - e.g. arithmetic
- This implies that any finite representation A of some infinte theory T such as arithmetic is incomplete,

For all models M of some theory T and $A \subset T$ is finite, $A \vdash s$ implies $M \models s$

Perhaps Some More Bad News



- Even if we accept the previous issues and continue to press on…
- …the proofs that some decision procedure is likely to construct are completely unstructured

```
1 y v x = x v y & (x v y) v z = x v (y v z) & ((x v y)' v (x' v y)')' = y # answer(robbins_basis) # label(non_clause) # label(goal).  [goal].
2 (((x v y)' v z)' v (x v (z' v (z v u)')')')' = z # label(DN1).  [assumption].
3 c1 v c2 != c2 v c1 | (c2 v c1) v c3 != c2 v (c1 v c3) | ((c2 v c1)' v (c2' v c1)')' != c1 # answer(robbins_basis).  [deny(1)].
4 c2 v c1 != c1 v c2 | (c2 v c1) v c3 != c2 v (c1 v c3) | ((c2 v c1)' v (c2' v c1)')' != c1 # answer(robbins_basis).  [copy(3),flip(a)].
5 ((x v y)' v (((z v u)' v x)' v (y' v (y v w)')')')' = y.  [para(2(a,1),2(a,1,1,1,1,1))].
18 ((x v x')' v x)' = x'.  [para(2(a,1),5(a,1,1,2))].
22 (x' v (x v (x' v (x v y)')')')' = x.  [para(18(a,1),2(a,1,1,1))].
27 ((x v y)' v (x' v (y' v (y v z)')')')' = y.  [para(22(a,1),2(a,1,1,1,1,1))].
31 (((x v y)' v z)' v (x v z)')' = z.  [para(22(a,1),2(a,1,1,2,1,2))].
58 ((x v y)' v (x' v y)')' = y.  [para(22(a,1),31(a,1,1,1,1,1))].
64 (x v ((y v z)' v (y v x)')')' = (y v x)'.  [para(31(a,1),31(a,1,1,1))].
65 (((((x v y)' v z)' v u)' v (x v z)')' v z)' = (x v z)'.  [para(31(a,1),31(a,1,1,2))].
66 c2 v c1 != c1 v c2 | (c2 v c1) v c3 != c2 v (c1 v c3) # answer(robbins_basis).  [back_rewrite(4),rewrite([58(29)]),xx(c)].
94 ((((x v (x v y)')' v z)' v x)' v (x v y)')' = x.  [para(58(a,1),2(a,1,1,2))].
101 (((x v x') v x)' v x')' = x.  [para(18(a,1),58(a,1,1,2))].
111 ((x v y)' v ((z v x)' v y)')' = y.  [para(58(a,1),31(a,1,1,1,1,1))].
112 (x v (y v (y' v x)')')' = (y' v x)'.  [para(58(a,1),31(a,1,1,1))].
…
6181 x v (y v z) = z v (y v x).  [para(6167(a,1),999(a,1,2)),rewrite([6179(3),796(4),6179(4)])].
6182 $F # answer(robbins_basis).  [resolve(6181,a,1138,a)].
```

source: prover9 proof archive

Some Successes

- Perhaps the most famous success in fully automatic theorem proving is the proof of the *Robbins Conjecture*:
  - A problem first posed by E.V.Huntington in 1933 and then refined by Herbert Robbins:

For all elements *a*, *b*, and *c*:

1. Associativity: $a \vee (b \vee c) = (a \vee b) \vee c$
2. Commutativity: $a \vee b = b \vee a$
3. *Robbins equation*: $\neg (\neg (a \vee b) \vee \neg (a \vee \neg b)) = a$

  - Are all Robbins algebras Boolean?
  - Yes! – proved by William McCune with the theorem prover EQP in 1996 – it took 172 hrs ≈ 1 week

source: http://www.cs.unm.edu/~mccune/papers/robbins/

Other Fully Automatic TPs

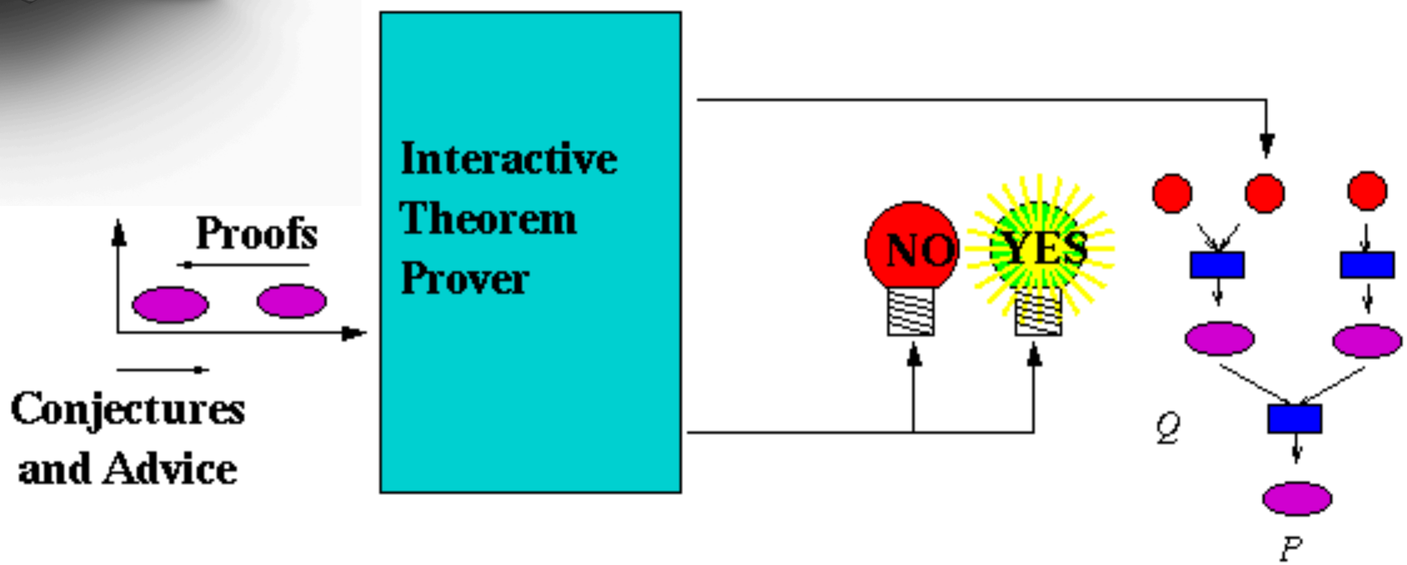
- E - http://wwwlehre.dhbw-stuttgart.de/~sschulz/E/E.html
- ACL2 - http://www.cs.utexas.edu/users/moore/acl2/
- Prover9 - http://www.cs.unm.edu/~mccune/prover9/
- Many others

Yet…

- After almost 50 years of research in fully automatic theorem proving the results are pretty thin…

- …perhaps a better strategy is a collaboration between proof author and automatic theorem prover.

Proof Assistants



Definition

- In computer science and mathematical logic, a *proof assistant* or *interactive theorem prover* is a software tool to assist with the development of formal proofs by human-machine collaboration. This involves some sort of interactive proof editor, or other interface, with which a human can guide the search for proofs, the details of which are stored in, and some steps provided by, a computer.

source: http://en.wikipedia.org/wiki/Proof_assistant

Proof Assistants

- proof assistants avoid decidability problems by relying on the human to structure the proof in such a way that only valid sentences need to validated.
- proof assistants avoid the NP problems because typically proofs are broken down into small steps that don't require a lot of search in order to be validated
- interesting ramification: logics used in proof assistants do *not* have to be complete!
  - the TP does not have to rely on the fact that everything that is true in the models can be proven
  - rather, we rely on the fact that the conclusion *follows* from the premises
  - this allows us to use much more powerful logics in proof assistants than would be possible in fully automatic theorem provers

The Mizar System

- Perhaps the oldest proof assistant – started in 1973 by Andrzej Trybulec.
- Based on first-order logic and set theory
- Very large library of existing proofs – as of 2012:
  - 1150 articles written by 241 authors
  - these contain more than 10,000 formal definitions of mathematical objects and about 52,000 theorems proved on these objects
  - some examples are: Hahn–Banach theorem, König's lemma, Brouwer fixed point theorem, Gödel's completeness theorem and Jordan curve theorem.

http://mizar.org/

A Simple Mizar Proof: √2 is irrational


```
theorem
  sqrt 2 is irrational
proof
 assume sqrt 2 is rational;
 then consider i being Integer, n being Nat such that
W1: n<>0 and
W2: sqrt 2=i/n and
W3: for i1 being Integer, n1 being Nat st n1<>0 & sqrt 2=i1/n1 holds n<=n1
      by RAT_1:25;
A5: i=sqrt 2*n by W1,XCMPLX_1:88,W2;
C: sqrt 2>=0 & n>0 by W1,NAT_1:19,SQUARE_1:93;
 then i>=0 by A5,REAL_2:121;
 then reconsider m = i as Nat by INT_1:16;
A6: m*m = n*n*(sqrt 2*sqrt 2) by A5
  .= n*n*(sqrt 2)^2 by SQUARE_1:def 3
  .= 2*(n*n) by SQUARE_1:def 4;
 then 2 divides m*m by NAT_1:def 3;
 then 2 divides m by INT_2:44,NEWTON:98;
 then consider m1 being Nat such that
W4: m=2*m1 by NAT_1:def 3;
 m1*m1*2*2 = m1*(m1*2)*2
    .= 2*(n*n) by W4,A6,XCMPLX_1:4;
 then 2*(m1*m1) = n*n by XCMPLX_1:5;
 then 2 divides n*n by NAT_1:def 3;
 then 2 divides n by INT_2:44,NEWTON:98;
 then consider n1 being Nat such that
W5: n=2*n1 by NAT_1:def 3;
A10: m1/n1 = sqrt 2 by W4,W5,XCMPLX_1:92,W2;
A11: n1>0 by W5,C,REAL_2:123;
  then 2*n1>1*n1 by REAL_2:199;
 hence contradiction by A10,W5,A11,W3;
end;
```

source: Freek Wiedijk’s book *The Seventeen Provers of the World*

The Coq System

- Started in 1984
- Implements a higher order logic: higher-order type theory
  - not complete and not decidable but sound
  - very expressive
- Coq is used in a large variety of domains such as formalization of mathematics, specification and verification of computer programs, etc.

source: https://coq.inria.fr

Example: √2 is irrational


```
Theorem irrational_sqrt_2: irrational (sqrt 2%nat).
intros p q H H0; case H.
apply (main_thm (Zabs_nat p)).
replace (Div2.double (q * q)) with (2 * (q * q));
 [idtac | unfold Div2.double; ring].
case (eq_nat_dec (Zabs_nat p * Zabs_nat p) (2 * (q * q))); auto; intros H1.
case (not_nm_INR _ _ H1); (repeat rewrite mult_INR).
rewrite <- (sqrt_def (INR 2)); auto with real.
rewrite H0; auto with real.
assert (q <> 0%R :> R); auto with real.
field; auto with real; case p; simpl; intros; ring.
Qed.
```

```
main_thm =
fun n : nat =>
lt_wf_ind n
  (fun n0 : nat => forall p : nat, n0 * n0 = Div2.double (p * p) -> p = 0)
  (fun (n0 : nat)
     (H : forall m : nat,
          m < n0 -> forall p : nat, m * m = Div2.double (p * p) -> p = 0)
     (p : nat) (H0 : n0 * n0 = Div2.double (p * p)) =>
   match Peano_dec.eq_nat_dec n0 0 with
   | left H1 =>
       let H2 :=
         eq_ind_r (fun n : nat => n * n = Div2.double (p * p) -> p = 0)
           match p as n return (0 * 0 = Div2.double (n * n) -> n = 0) with
           | O => fun H2 : 0 * 0 = Div2.double (0 * 0) => H2
           | S n0 =>
               fun H2 : 0 * 0 = Div2.double (S n0 * S n0) =>
               let H3 :=
                 eq_ind (0 * 0)
                   (fun ee : nat =>
                    match ee with
                    | O => True
                    | S _ => False
                    end) I (Div2.double (S n0 * S n0)) H2 in
               False_ind (S n0 = 0) H3
           end H1 in
       H2 H0
   | right H1 => ....
```

Isabelle

- Isabelle is a proof assistant which implements higher-order logic:
  - LCF – lambda calculus extended with logical constructs
  - incomplete, undecidable, but sound
- Isabelle is developed at University of Cambridge (Larry Paulson), Technische Universität München (Tobias Nipkow) and Université Paris-Sud (Makarius Wenzel).
- The main application is the formalization of mathematical proofs and in particular *formal verification*, which includes proving the correctness of computer hardware or software and proving properties of computer languages and protocols.

source: http://isabelle.in.tum.de

Example: √2 is irrational


```
theorem sqrt2_not_rational:
  "sqrt (real 2) ∉ ℚ"
proof
  assume "sqrt (real 2) ∈ ℚ"
  then obtain m n :: nat where
    n_nonzero: "n ≠ 0" and sqrt_rat: "¦sqrt (real 2)¦ = real m / real n"
    and lowest_terms: "gcd m n = 1" ..
  from n_nonzero and sqrt_rat have "real m = ¦sqrt (real 2)¦ * real n" by simp
  then have "real (m²) = (sqrt (real 2))² * real (n²)" by (auto simp add: power2_eq_square)
  also have "(sqrt (real 2))² = real 2" by simp
  also have "... * real (m²) = real (2 * n²)" by simp
  finally have eq: "m² = 2 * n²" ..
  hence "2 dvd m²" ..
  with two_is_prime have dvd_m: "2 dvd m" by (rule prime_dvd_power_two)
  then obtain k where "m = 2 * k" ..
  with eq have "2 * n² = 2² * k²" by (auto simp add: power2_eq_square mult_ac)
  hence "n² = 2 * k²" by simp
  hence "2 dvd n²" ..
  with two_is_prime have "2 dvd n" by (rule prime_dvd_power_two)
  with dvd_m have "2 dvd gcd m n" by (rule gcd_greatest)
  with lowest_terms have "2 dvd 1" by simp
  thus False by arith
qed
```

Observations

- Pros:
  - Powerful reasoning mechanisms – deduction, induction, tactics, etc
  - Expressive proof languages
- Cons:
  - steep learning curve for the systems
  - the complicated proof languages represent an adoption hurdle

Prolog as a Proof Assistant

- I am interested in ATP coming from a formal semantics for programming languages angle:
  - build programming language models
  - reason about these models

Prolog as a Proof Assistant

- I needed the following:
  - a language that can serve both as a specification language and a language to reason about specifications
  - a language is easy to learn
    - simple first-order logic
    - modus ponens as the main deduction mechanism
  - robust implementation
    - something that does not feel like a graduate student project ☺

Prolog as Proof Assistant

- Prolog fits the bill
  - designed as a programming language
  - rigorously based on first-order logic
  - uses a resolution based deduction engine (think automated modus ponens)
  - easy to learn
  - ISO standarized
  - lots of commercial and open source implementations available
  - I use SWI Prolog (www.swi-prolog.org)

Prolog as a Proof Assistant

- Downside:
  - no equational reasoning
    - writing a proof that $\sqrt{2}$ is irrational is difficult in Prolog
  - no type system
    - will not catch typos in term structures – difficult debugging

Prolog – A Simple Program


```
% facts
female(betty).
male(bob).
parent(betty,bob).

% rule
mother(X,Y) :- parent(X,Y),female(X).

% query
:- mother(Q,bob).
```

$\forall X, Y[\text{mother}(X, Y) \text{ if } \text{parent}(X, Y) \wedge \text{female}(X)]$

$\exists Q[\text{mother}(Q,\text{bob})]$

You just learned 90% of the Prolog language!

Prolog – Another Program


```
% recursive counting of elements
% in a list.

% base case:
% the count of an empty list is 0
count([ ],0).

% recursive step:
% the count of any list List is Count if
%     List can be divided into a First element and the Rest of the list and
%     T is the count of the Rest of the list and
%     Count is T plus 1.
count(List,Count) :-
        List=[ First | Rest ],
        count(Rest,T),
        Count is T + 1.

% try it!
:- count([1,2,3],P),writeln(P).
```

Prolog as a Theorem Prover

- We have developed a library that makes Prolog deductions sound but incomplete
  - This is OK because we are using it as a proof assistant – only soundness is required.
  - interesting side node – with a little bit of work Prolog could be made *quasi-complete*
- Our library makes Prolog easy to use as a TP

Semantic Specifications 101

- We will define a simple calculator like language
- build a first-order logic model
- and then reason about the model

Semantic Specifications 101


```
% syntax definition -- Lisp like prefix notation for expressions
%
% syntax of expressions
%
%  E ::= X
%     | L
%     | mult(E,E)
%     | plus(E,E)
%     | minus(E,E)
%
% syntax of statements
%
%  S ::= assign(X,E)
%     | print(E)
%     | S @ S
%
%  L ::= <any integer digit>
%  X ::= <any variable name>
```

Example: assign(x,plus(10,1)) @ print(x)

Semantic Specifications 101


```
% semantic definition of integer expressions

L -->> L :-
        is_int(L),!.

B:: X -->> V :-
        is_var(X),
        lookup(X,B,V),!.

B:: mult(E1,E2) -->> V :-
        B:: E1 -->> V1,
        B:: E2 -->> V2,
        V xis V1 * V2,!.

B:: plus(E1,E2) -->> V :-
        B:: E1 -->> V1,
        B:: E2 -->> V2,
        V xis V1 + V2,!.

B:: minus(E1,E2) -->> V :-
        B:: E1 -->> V1,
        B:: E2 -->> V2,
        V xis V1 - V2,!.
```

- A simple ‘abstract interpreter’ model
- The main operator in a semantic specification is the ‘maps to’ operator -->>
- This operator maps a piece of syntax into its semantic domain (under the possible context of a state – the ‘::’ part)

Semantic Specifications 101


```
% semantic definition of statements

B:: assign(X,E) -->> [ (X,V) | B ] :-
        is_var(X),
        B:: E -->> V,!.

B:: print(E) -->> B :-
        B:: E -->> V,
        write('Output value: '),
        writeln(V),!.

B:: S1 @ S2 -->> B2 :-
         B:: S1 -->> B1,
        B1:: S2 -->> B2,!.
```

- Given a state the semantic value of a statement is another state!

That's it!

```
% for convenience make '@' infix and left associative
:- op(725,yfx,@).
```

Running a Calc Program


```
Welcome to SWI-Prolog (Multi-threaded, 64 bits, Version 6.6.6)
Copyright (c) 1990-2013 University of Amsterdam, VU Amsterdam
SWI-Prolog comes with ABSOLUTELY NO WARRANTY. This is free software,
and you are welcome to redistribute it under certain conditions.
Please visit http://www.swi-prolog.org for details.

For help, use ?- help(Topic). or ?- apropos(Word).

?- consult('calc-sem.pl').
%   xis.pl compiled 0.00 sec, 33 clauses
%  preamble.pl compiled 0.01 sec, 45 clauses
% calc-sem.pl compiled 0.01 sec, 58 clauses
true.

?- s:: assign(x,plus(10,1)) @ print(x) -->> S.
Output value: 11
S = [ (x, 11)|s].

?-
```

Proof – Semantic Equivalence


```
:- >>> 'Show that mult(2,3) is semantically equiv to add(3,3),'.
:- >>> 'it suffices to show that'.
:- >>> '   (forall s)(exists V)'.
:- >>> '        [s:: mult(2,3)-->>V ^ s:: plus(3,3)-->>V]'.

% proof
:- show s:: mult(2,3)-->>V , s:: plus(3,3)-->>V.
```

Proof – All Programs Terminate

- This is obvious because our language does not have function calls or loops
- but it still nice to actually prove it!
- The proof will show that the execution of any and every program will result in a value.
- Because syntactic domains can be viewed as inductively defined sets we can use induction to prove this.

Proof – All Programs Terminate

- Recall our syntax:

```
% E ::= X
%    | L
%    | mult(E,E)
%    | plus(E,E)
%    | minus(E,E)
%
% S ::= assign(X,E)
%    | print(E)
%    | S @ S
%
% L ::= <any integer digit>
% X ::= <any variable name>
```

Proof – All Programs Terminate


```
:- >>> 'induction on expressions'.

:- >>> 'Base cases:'.

:- >>> 'Variables'.
:- >>> 'Assume that states are finite'.
:- assume lookup(x,s,vx).
:- show s:: x -->> vx.

:- >>> 'Constants'.
:- assume is_int(n).
:- show s:: n -->> n.

:- >>> 'Inductive cases'.

:- >>> 'Operators'.
:- >>> 'mult'.
:- assume s:: a -->> va.
:- assume s:: b -->> vb.
:- show s:: mult(a,b) -->> va*vb.

:- >>> 'the remaining operators'.
:- >>> 'can be proved similarly'.
```

```
:- >>> 'induction on programming constructs'.

:- >>> 'Base cases:'.

:- >>> 'assignments'.
:- assume s:: e -->> ve.
:- show s:: assign(x,e) -->> [(x,ve)|s].

:- >>> 'print'.
:- assume s:: e -->> ve.
:- show s:: print(e) -->> s.

:- >>> 'Inductive step:'.

:- >>> 'composition'.
:- assume _A:: s1 -->> v1.
:- assume _B:: s2 -->> v2.
:- show s:: s1 @ s2 -->> v2.
```

Conclusions

- Fully automatic TP seems to be doomed because of the semi-decidability and NP trap
- Collaborative ATP or Proof Assistants build on the strengths of the structured approach humans take to theorem proving
- Collaborative ATP or Proof Assistants are versatile; going beyond mathematical theorem proving -- we have hardware verification, programming language semantics, etc.
- We are interested in Prolog as a theorem prover because of its simplicity, robustness, and availability – easy to learn – interesting as a first step into the theorem proving arena

Shameless Plug

- If you are interested in a mathematical approach to programming languages and theorem proving…
- …I teach a course in programming language semantics which applies some of the things we saw here and more – CSC501

Thank You!

- Slides and Prolog code available at my homepage :
  - http://homepage.cs.uri.edu/faculty/hamel/

(under publications in the talks section)