We[1] will use first order logic as the basis for our reasoning. Without going into the formal details of first order logic terminology and sentence construction we have the following statements:

- $A \wedge B$ denotes the conjunction A and B,
- $A \vee B$ denotes the disjunction A or B,
- $\neg A$ denotes the negation not A,
- $A \Rightarrow B$ denotes the implication, if A then B,
- $A \Leftrightarrow B$ denotes the logical equivalence, A if and only if B (often written as A iff B),

where A and B are statements or assertions.

[1] The material presented here is based on "Naive Set Theory" by P. Halmos and "The Formal Semantics of Programming Languages" by G. Winskel.

Observe the precedences of the logical operators, ordered from high to low:

- $\neg$
- $\wedge, \vee$
- $\Rightarrow, \Leftrightarrow$

A Word or Two about Implication

The truth table for the implication operator '$\Rightarrow$' can be given as

	A	B	$A \Rightarrow B$
(1)	1	0	0
(2)	1	1	1
(3)	0	0	1
(4)	0	1	1

Entries (1) and (2) are intuitive: When the antecedent A is true but the consequent B is false then the implication itself is false. If both the antecedent and the consequent are true then the implication is true.

However, entries (3) and (4) are somewhat counter intuitive. They state that if the antecedent A is false then the implication is true regardless of the value of the consequent. In other words, we can conclude "anything" from an antecedent that is false. In mathematical jargon we say that (3) and (4) **hold trivially**.

A Word or Two about Implication – An Example

If Bob is a bachelor, then he is single.
Bob is a bachelor.

∴ Bob is single.

Now consider an antecedent that is not true,

If Bob is a bachelor, then he is single.
Bob is not a bachelor.

∴ Bob is not single (by rule (3)).

Since the antecedent is not true rule (3) allows us to conclude the opposite of what the implication dictates. However, the following is also valid reasoning,

If Bob is a bachelor, then he is single.
Bob is not a bachelor.

∴ Bob is single (by rule (4)).

Not being a bachelor does not necessarily imply that Bob is not single. For example, Bob could be a widower or a divorcee.

A Word or Two about Implication

Given the truth table for implication,

	A	B	$A \Rightarrow B$
(1)	1	0	0
(2)	1	1	1
(3)	0	0	1
(4)	0	1	1

this means that in order to show that an implication holds we only have to show that rule (2) holds. Rule (1) states that the implication is false and rules (3) and (4) are trivially true and therefore not interesting.

Closely Related: Equivalence

We write $A \Leftrightarrow B$ if A and B are equivalent.
Given the truth table for the equivalence operator is given as,

	A	B	$A \Leftrightarrow B$
(1)	1	0	0
(2)	1	1	1
(3)	0	0	1
(4)	0	1	0

That is, the operator only produces a true value if A and B have the same truth assignment.
Another, and very useful, way to look at the equivalence operator is as follows:

$$A \Leftrightarrow B \equiv A \Rightarrow B \wedge B \Rightarrow A$$

Exercise: Construct the above truth table using this definition of the equivalence operator.

We also allow predicates (properties) as part of our notation,

$$P(x)$$

where the predicate P is true if it holds for x otherwise it is false. We view our standard relational operators as binary predicates. For example, the predicate $P(x)$ that expresses the fact that x is less or equal to 3 is written as,

$$P(x) \equiv x \leq 3.$$

Note: Predicates can have arities larger than 1, e.g. $P(x, y)$ with $P(x, y) \equiv x \leq y$.

We also allow for the quantifiers $\exists$ (there exists) and $\forall$ (for all) in our logical statements,

- $\exists x.\, P(x)$ – "there exists an x such that $P(x)$"
- $\forall x.\, P(x)$ – "for all x such that $P(x)$"

Some examples,

- $\forall x, \exists y.\, y = x^2$
- $\forall x, \forall y.\, \text{female}(x) \wedge \text{child}(x, y) \Rightarrow \text{mother}(x, y)$

Sets[2] are unordered collections of objects and are usually denoted by capital letters. For example, let a, b, c denote some objects then the set A of these objects is written as,

$$A = \{a, b, c\}.$$

There are a number of standard sets which come in handy,

- $\emptyset$ denotes the empty set, i. e. $\emptyset = \{\}$,
- $\mathbb{N}$ denotes the set of all natural numbers including 0, e. g. $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$,
- $\mathbb{I}$ denotes the set of all integers, $\mathbb{I} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$,
- $\mathbb{R}$ denotes the set of all reals,
- $\mathbb{B}$ denotes the set of boolean values, $\mathbb{B} = \{\mathit{true}, \mathit{false}\}$.

[2] Read Sections 2.1 and 2.2 in the book by David Schmidt.

The most fundamental property in set theory is the notion of *belonging*,

$$a \in A \text{ iff } a \text{ is an element of the set } A.$$

The notion of belonging allows us to define *subsets*,

$$Z \subseteq A \text{ iff } \forall e \in Z \,.\, e \in A.$$

We define set *equivalence* as,

$$A = B \text{ iff } A \subseteq B \wedge B \subseteq A$$

We can construct new sets from given sets using *union*,

$$A \cup B = \{e \mid e \in A \vee e \in B\},$$

and *intersection*,

$$A \cap B = \{e \mid e \in A \wedge e \in B\}.$$

There is another important set construction called the *cross product*,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\},$$

$A \times B$ is the set of all ordered pairs where the first component of the pair is drawn from the set A and the second component of the pair is drawn from B. (
Exercise: Let $A = \{a, b\}$ and $B = \{c, d\}$, construct the set $A \times B$.

A construction using subsets is the *powerset* of some set X,

$$\mathcal{P}(X),$$

The *powerset* of set X is set of all subsets of X. For example, let $X = \{a, b\}$, then

$$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Note: $\emptyset \subset X$

Exercise: What would $\mathcal{P}(X \times X)$ look like?

The fact that $\emptyset \subset X$ for any set X is interesting in its own right. Let's see if we can prove it.

Proof: Proof by contradiction. Assume X is any set. Assume that $\emptyset$ is not a subset of X. Then the definition of subsets,

$$A \subseteq B \Leftrightarrow \forall e \in A . e \in B,$$

implies that there exist at least one element in $\emptyset$ that is not also in X. But that is not possible because $\emptyset$ has no elements – a contradiction. Therefore, our assumption the that $\emptyset$ is not a subset of X must be wrong and we can conclude that $\emptyset \subset X$.

A (binary) relation is a set of ordered pairs. If R is a relation that relates the elements of set A to the elements B, then

$$R \subseteq A \times B.$$

This means if $a \in A$ is related to $b \in B$ via the relation R, then $(a, b) \in R$. We often write

$$a \, R \, b.$$

Consider the relational operator $\leq$ applied to the set $\mathbb{N} \times \mathbb{N}$. This induces a relation, call it $\leq \subseteq \mathbb{N} \times \mathbb{N}$, with $(a, b) \in \leq$ (or $a \leq b$ in our relational notation) if $a \in \mathbb{N}$ is less or equal to $b \in \mathbb{N}$.

The first and second components of each pair in some relation R are drawn from different sets called the *projections* of R onto the first and second *coordinate*, respectively. We introduce the operators *domain* and *range* to accomplish these projections. Let $R \subseteq A \times B$, then,

$$\mathrm{dom}(R) = A,$$

and

$$\mathrm{ran}(R) = B.$$

In this case we talk about a relation *from A to B*. The range is often called the co-domain. If $R \subseteq X \times X$, then

$$\mathrm{dom}(R) = \mathrm{ran}(R) = X.$$

Here we talk about a relation *in X*.

Let $R \subseteq X \times X$ such that $(a, b) \in R$ iff $a = b$. That is, R is the *equality relation* in X. (What do the elements of the equality relation look like for $\mathbb{N} \times \mathbb{N}$?)

A relation $R \subseteq X \times X$ is an *equivalence relation* if the following conditions hold,

- R is *reflexive*[3] – $x\,R\,x$,
- R is *symmetric* – $x\,R\,y \Rightarrow y\,R\,x$,
- R is *transitive* – $x\,R\,y \wedge y\,R\,z \Rightarrow x\,R\,z$,

where $x, y, z \in X$.

The *smallest* equivalence relation in some set X is the equality relation defined above. The *largest* equivalence relation is some set is the cross product $X \times X$. (Consider the smallest/largest equiv. relation in $\mathbb{I}$)

[3] Recall that $x\,R\,x \equiv (x, x) \in R$

A *function* f from X to Y is a relation $f \subseteq X \times Y$ such that

$$\forall x \in X, \exists y, z \in Y. (x, y) \in f \land (x, z) \in f \Rightarrow y = z.$$

In other words, each $x \in X$ has a unique value $y \in Y$ with $(x, y) \in f$ or functions are constrained relations.

We let $X \to Y$ denote the *set of all functions* from X to Y (i. e. $X \to Y \subset \mathcal{P}(X \times Y)$, why is the subset strict? Hint: it is not a relation), then the customary notation for specifying functions can be defined as follows,

$$f : X \to Y \text{ iff } f \in X \to Y.$$

For *function application* it is customary to write

$$f(x) = y$$

for $(x, y) \in f$. In this case we say that the function is *defined* at point x. Otherwise we say that the function is *undefined* at point x and we write $f(x) = \perp$.

Note that $f(\perp) = \perp$ and we say the f is *strict*.

We say that $f : X \to Y$ is a *total* function if f is defined for all $x \in X$. Otherwise we say that f is a *partial* function.

We can now make the notion of a predicate formal – a predicate is a function whose range (co-domain) is restricted to the boolean values:

$$P : X \to \mathbb{B}$$

where P is a predicate that returns true or false for the objects in set X.

Example: Let U be the set of all possible objects – a universe if you like, and let,

$$human : U \to \mathbb{B}$$

be the predicate that returns true if the object is a human and will return false otherwise, then

$$human(\text{socrates}) = true$$
$$human(\text{car}) = false$$

1. In your own words explain what the function $m : X \times Y \to Z$ does.
2. How would you describe the function $c : X \to (Y \to Z)$?
3. In your own words explain what the relation $R \subseteq (X \times Y) \times (Z \times W)$ does.